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AN INTRINSIC THEORY OF A COSSERAT CONTINUUM

YOSHIYUKI YAMAMOTO

Faculty of Engineering, University of Tokyo, Bunkyo-ku, Tokyo

Abstract—In this paper, a linear intrinsic theory is developed for a Cosserat continuum as a natural extension of the classical theory of elasticity. Introducing the stress functions, a decomposition theorem of Hilbert space for stress distribution is derived, and the properties of the so-called eigen stress is discussed.

1. INTRODUCTION

MANY papers have been published for the elasticity of Cosserat continuum, and some of them have treated the fundamental characteristics of this new elasticity [1–6]. In this paper, stress functions are introduced naturally from the global condition of equilibrium, and the condition of compatibility is investigated thoroughly. The structure of Hilbert space in which the stress distributions are expressed is examined, and a decomposition theorem is proved by a method similar to that of the classical theory of elasticity [7].

Assume a smooth surface in a Cosserat continuum in a state. After deformation, however, this surface becomes jagged, and it is due to the deformations of the micro-medium. As to a deformed Cosserat continuum, it is assumed that the micro-rotation or the rotation of the micro-medium relative to the macro-medium is due to the so-called couple-stress, which contributes a part of the moment exerting on an inner surface of a Cosserat continuum.

Let x^i, $(i = 1, 2, 3)$, be the rectangular Cartesian coordinate. Assume that a Cosserat continuum locates in a certain domain V. The coordinate x^i of a material point can be introduced in the undeformed configuration, and it is regarded as the Lagrangian coordinate. In this paper, the summation convention is used as usual, and indices after a comma indicate the ordinary differentiation with respect to appropriate Lagrangian coordinates.

2. STRESS FUNCTIONS

In a deformed state, the stress distribution s^{ij} in the micro-medium is not necessarily smooth but may be singular. From the view-point of the macro-medium, the stress distribution s^{ij} can be equivalently substituted by the macro-stress σ^{ij} and the couple-stress $\mu^{ijk}(= -\mu^{jik})$:

$$\begin{aligned} \iint_X s^{ij}n_j \,\mathrm{d}X &= \iint_X \sigma^{ij}n_j \,\mathrm{d}X \\ \iint_X x^{[j}s^{i]k}n_k \,\mathrm{d}X &= \iint_X (x^{[j}\sigma^{i]k} + \mu^{ijk})n_k \,\mathrm{d}X \end{aligned} \tag{1}$$

where X is any smooth surface portion in the undeformed state, whose diameter is larger than the scale of the micro-medium, n_j is the unit normal vector on it, and brackets used with indices mean the mixing with respect to appropriate indices:

$$x^{[j}s^{i]k} = \tfrac{1}{2}(x^j s^{ik} - x^i s^{jk}) \tag{2}$$

The macro-stress σ^{ij} and the couple-stress μ^{ijk}, which will be referred as the stress (σ, μ), can be considered to be expressed by differentiable functions of x^i.

Consider a self-equilibrated stress (σ, μ). The corresponding resultant force and moment on any two surfaces with a common boundary curve give the same values, owing to equilibrium of the body portion between these two surfaces. Therefore, the resultant force and moment on any surface region X should be determined by line integrals of certain functions on the boundary ∂X of X:

$$\begin{aligned} \iint_X \sigma^{ij} n_j \, \mathrm{d}X &= \tfrac{1}{2}\int_{\partial X} \phi^{ijk} e_{jkp} \, \mathrm{d}x^p \\ \iint_X (x^{[j}\sigma^{i]k} + \mu^{ijk}) n_k \, \mathrm{d}X &= \tfrac{1}{2}\int_{\partial X} \phi_0^{ijkl} e_{klp} \, \mathrm{d}x^p \end{aligned} \tag{3}$$

where

$$e_{ijk} = e^{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is an even permutation of 123} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The functions ϕ^{ijk} and ϕ_0^{ijkl} are considered as such differentiable functions of x^i that

$$\phi^{ijk} = -\phi^{ikj}, \qquad \phi_0^{ijkl} = -\phi_0^{jikl} = -\phi_0^{ijlk}. \tag{5}$$

By virtue of Stokes' theorem, (3) becomes as

$$\begin{aligned} \iint \sigma^{ij} n_j \, \mathrm{d}X &= \iint \phi^{ijk}{}_{,k} \, n_j \, \mathrm{d}X \\ \iint (x^{[j}\sigma^{i]k} + \mu^{ijk}) n_k \, \mathrm{d}X &= \iint \phi_0^{ijkl}{}_{,l} \, n_k \, \mathrm{d}X. \end{aligned} \tag{6}$$

It then follows that

$$\sigma^{ij} = \phi^{ijk}{}_{,k} \tag{7a}$$

$$x^{[j}\sigma^{i]k} + \mu^{ijk} = \phi_0^{ijkl}{}_{,l}. \tag{8}$$

Eliminating ϕ^{ijk} and ϕ_0^{ijkl} from (7a) and (8), we have the equilibrium condition in the absence of body forces and body couples

$$\left.\begin{aligned} \sigma^{ij}{}_{,j} &= 0 \\ \sigma^{[ij]} + \mu^{ijk}{}_{,k} &= 0 \end{aligned}\right\} \text{ in } V. \tag{9}$$

It is obvious that (9) is the condition of integrability [8] for the solutions ϕ^{ijk} and ϕ_0^{ijkl} of (7a) and (8). It follows from (7a) and (8) that

$$\mu^{ijk} = -\phi^{[ij]k} + \gamma^{ijkl}{}_{,l} \tag{7b}$$

where

$$\gamma^{ijkl} = \phi_0^{ijkl} - x^{[j}\phi^{i]kl} = -\gamma^{jikl} = -\gamma^{ijlk}. \tag{10}$$

Any self-equilibrated stress can be expressed in the form of (7) with certain functions ϕ^{ijk} and γ^{ijkl}. The functions ϕ^{ijk} and γ^{ijkl} are similar to the stress functions of Maxwell and Morrera for the classical theory of elasticity [7]. By virtue of (7), (6) can be rewritten as

$$\iint_X \sigma^{ij} n_j \, \mathrm{d}X = \tfrac{1}{2}\int_{\partial X} \phi^{ijk} e_{jkp} \, \mathrm{d}x^p \tag{11a}$$

$$\begin{aligned}\iint_X &(x^{[j}\sigma^{i]k} + \mu^{ijk}) n_k \, \mathrm{d}X \\ &= \iint_X (x^{[j}\phi^{i]kl}{}_{,l} - \phi^{[ij]k} + \gamma^{ijkl}{}_{,l}) n_k \, \mathrm{d}X \\ &= \tfrac{1}{2}\int_{\partial X} (x^{[j}\phi^{i]kl} + \gamma^{ijkl}) e_{klp} \, \mathrm{d}x^p.\end{aligned} \tag{11b}$$

3. CONSTITUTIVE EQUATIONS AND HILBERT SPACE FOR STRESS

It can be assumed that the strain energy density $U(\sigma, \mu)$ due to the stress (σ, μ) is given by the quadratic form

$$U(\sigma, \mu) = \tfrac{1}{2}E_{ijpq}\sigma^{ij}\sigma^{pq} + F_{ijpqr}\sigma^{ij}\mu^{pqr} + \tfrac{1}{2}G_{ijkpqr}\mu^{ijk}\mu^{pqr} \tag{12}$$

where

$$\begin{aligned} E_{ijpq} &= E_{pqij}, \qquad F_{ijpqr} = -F_{ijqpr} \\ G_{ijkpqr} &= G_{pqrijk} = -G_{jikpqr} = -G_{ijkqpr}. \end{aligned} \tag{13}$$

The quadratic form $U(\sigma, \mu)$ may be considered to be positive-definite. Corresponding to (σ, μ), the strain ε_{ij}, $\kappa_{ijk}\,(= -\kappa_{jik})$ can be defined as follows;

$$\begin{aligned} \varepsilon_{ij} &= \frac{\partial U}{\partial \sigma^{ij}} = E_{ijpq}\sigma^{pq} + F_{ijpqr}\mu^{pqr} \\ \kappa_{ijk} &= \frac{\partial U}{\partial \mu^{ijk}} = F_{pqijk}\sigma^{pq} + G_{pqrijk}\mu^{pqr}. \end{aligned} \tag{14}$$

Now the inner product of (σ, μ) and (σ^*, μ^*) can be defined in the form

$$\begin{aligned}(\sigma, \mu; \sigma^*, \mu^*) &= \varepsilon_{ij}\sigma^{*ij} + \kappa_{ijk}\mu^{*ijk} \\ &= E_{ijpq}\sigma^{ij}\sigma^{*pq} + F_{ijpqr}(\sigma^{ij}\mu^{*pqr} + \mu^{pqr}\sigma^{*ij}) + G_{ijkpqr}\mu^{pqr}\mu^{*ijk}.\end{aligned} \tag{15}$$

Inversely, the stress can be expressed in terms of the strain in the following form;

$$\begin{aligned} \sigma^{ij} &= E^{ijpq}\varepsilon_{pq} + F^{ijpqr}\kappa_{pqr} \\ \mu^{ijk} &= F^{pqijk}\varepsilon_{pq} + G^{pqrijk}\kappa_{pqr} \end{aligned} \tag{16}$$

where E^{ijpq}, F^{ijpqr}, and G^{ijkpqr} are given in terms of E_{ijpq}, F_{ijpqr}, and G_{ijkpqr}.

Hereafter Hilbert space $\mathfrak{H}$ will be introduced to express the stress (σ, μ) as a point in it. This Hilbert space can be defined by the inner product (15): The length of the radial vector (σ, μ) in $\mathfrak{H}$ is given by

$$(\sigma, \mu\,; \sigma, \mu)^{\frac{1}{2}}$$

and the orthogonality condition of the vectors (σ, μ) and (σ^*, μ^*) is given by

$$(\sigma, \mu\,; \sigma^*, \mu^*) = 0. \tag{17}$$

4. CONDITION OF COMPATIBILITY

The totality of the points in $\mathfrak{H}$ corresponding to such self-equilibrated stresses that

$$\begin{aligned} &\sigma^{ij} = \phi^{ijk}{}_{,k}, \qquad \mu^{ijk} = -\phi^{[ij]k} + \gamma^{ijkl}{}_{,l}\,; \\ &\phi^{ijk} = 0, \qquad \gamma^{ijkl} = 0 \quad \text{on } \partial V \end{aligned} \tag{18}$$

forms a subspace $\mathfrak{N}$ of $\mathfrak{H}$. Applying (11) on ∂V, it can be seen that the stress (σ, μ) in $\mathfrak{N}$ satisfies the condition

$$\left.\begin{aligned} \sigma^{ij}n_j &= 0 \\ (x^{[j}\sigma^{i]k} + \mu^{ijk})n_k &= 0 \end{aligned}\right\} \quad \text{on } \partial V. \tag{19}$$

These are considered as the equilibrium condition on ∂V in the absence of external forces.

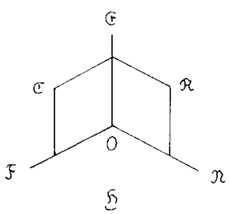

FIG. 1. Hilbert space $\mathfrak{H}$ for stress distributions.

The subspace $\mathfrak{C}$ of $\mathfrak{H}$ orthogonal to $\mathfrak{N}$ is the totality of such stresses (σ, μ) that

$$(\sigma, \mu\,; \sigma^*, \mu^*) = 0 \tag{20}$$

where (σ^*, μ^*) is any stress in $\mathfrak{N}$. This condition can be rewritten as

$$\iiint_V [\varepsilon_{ij}\phi^{*ijk}{}_{,k} + \kappa_{ijk}(-\phi^{*[ij]k} + \gamma^{*ijkl}{}_{,l})]\,\mathrm{d}V$$

$$= -\iiint_V [(\varepsilon_{i[j,k]} + \kappa_{i[jk]})\phi^{*ijk} + \kappa_{ij[k,l]}\gamma^{*ijkl}]\,\mathrm{d}V = 0$$

where ε_{ij} and κ_{ijk} correspond to (σ, μ) and ϕ^{*ijk} and γ^{*ijkl} to (σ^*, μ^*). It then follows that ε_{ij} and κ_{ijk} should satisfy the equations

$$\begin{aligned}\varepsilon_{i[j,k]}+\kappa_{i[jk]} &= 0\\ \kappa_{ij[k,l]} &= 0.\end{aligned} \tag{21}$$

From the second condition of (21), it can be shown that such a function $\psi_{ij}(= -\psi_{ji})$ exists that

$$\kappa_{ijk} = \psi_{ij,k}. \tag{22}$$

Substituting this into the first condition of (21), we have

$$(\varepsilon_{i[j}+\psi_{i[j}),_{k]} = 0.$$

Hence we can see that such a function u_i exists that

$$\varepsilon_{ij}+\psi_{ij} = u_{i,j} \tag{23a}$$

or

$$\varepsilon_{ij} = u_{i,j}-\psi_{ij}. \tag{23b}$$

The functions u_i and ψ_{ij} are regarded as the macro-displacement and the micro-rotation, respectively. Hence the above equations (21) are the condition of compatibility in the small. Integrating (22) and (23a), we can determine u_i and ψ_{ij} in the form

$$\psi_{ij}(P) = \psi_{ij}(A)+\int_A^P \kappa_{ijk}\,\mathrm{d}x^k \tag{24a}$$

$$\begin{aligned}u_i(P) &= u_i(A)+\int_A^P (\varepsilon_{ij}+\psi_{ij})\,\mathrm{d}x^j\\ &= u_i(A)+\psi_{ij}(x^j-x^j(A))\\ &\quad+\int_A^P [\varepsilon_{ik}-(x^j-x^j(A))\kappa_{ijk}]\,\mathrm{d}x^k\end{aligned} \tag{24b}$$

where A and P are a fixed and a generic point in V.

To obtain the condition that u_i and ψ_{ij} are determined uniquely in V, the topological property of the body V should be investigated [7]. Let p be the one-dimensional Betti number of V, and let $D_\alpha(\alpha = 1, \ldots, p)$ be such surfaces that V becomes a simply-connected domain by cutting along D_α. The one-dimensional Betti number of ∂V is $2p$, and ∂V can be

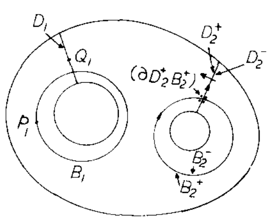

FIG. 2. Topology of V.

made a certain number of simply-connected surfaces by being cut along ∂D_α and appropriate closed curves B_α, $(\alpha = 1, \ldots, p)$. Without loss of generality, it can be assumed that each B_α does not meet any D_α's other than the corresponding one. Let $[\partial V]$ be the totality of such simply-connected surfaces.

Let X be a simply-connected surface in V. It then follows that

$$\oint_{\partial X} [\varepsilon_{ik} - (x^j - x^j(A))\kappa_{ijk}]\,\mathrm{d}x^k$$

$$= \tfrac{1}{2}\iint_X [\varepsilon_{i[k,l]} - \kappa_{i[lk]} - (x^j - x^j(A))\kappa_{ij[k,l]}e^{klp}n_p\,\mathrm{d}X = 0$$

$$\oint_{\partial X} \kappa_{ijk}\,\mathrm{d}x^k = \tfrac{1}{2}\iint_X \kappa_{ij[k,l]}e^{klp}n_p\,\mathrm{d}X = 0.$$

The quantities

$$\begin{aligned} \xi_i(B_\alpha) &= \oint_{B_\alpha} [\varepsilon_{ik} - x^j\kappa_{ijk}]\,\mathrm{d}x^k \\ \omega_{ij}(B_\alpha) &= \oint_{B_\alpha} \kappa_{ijk}\,\mathrm{d}x^k \end{aligned} \tag{25}$$

are regarded as the dislocation tensors, and the integrals

$$\oint [\varepsilon_{ij} - (x^j - x^j(A))\kappa_{ijk}]\,\mathrm{d}x^k$$

and

$$\oint \kappa_{ijk}\,\mathrm{d}x^k$$

taken for any closed path are given by a linear combination of $\xi^i(B_\alpha)$ or $\omega^{ij}(B_\alpha)$, respectively. Therefore, when, and only when,

$$\xi^i(B_\alpha) = 0, \qquad \omega^{ij}(B_\alpha) = 0 \tag{26}$$

u_i and ψ_{ij} given by (24) are determined uniquely, and (26) are the condition of compatibility in the large.

5. DECOMPOSITION THEOREM

Let $\mathfrak{F}$ be a subspace of $\mathfrak{H}$ corresponding to the totality of such stresses that

$$\begin{aligned} \sigma^{ij} &= E^{ijpq}(u_{p,q} - \psi_{pq}) + F^{ijpqr}\psi_{pq,r} \\ \mu^{ijk} &= F^{pqijk}(u_{p,p} - \psi_{pq}) + G^{pqrijk}\psi_{pq,r} \end{aligned} \tag{27}$$

where u_i and ψ_{ij} are one-valued functions. It is obvious that $\mathfrak{F}$ is a subspace of $\mathfrak{C}$. Let us consider the subspace $\mathfrak{R}$ of $\mathfrak{H}$ orthogonal to $\mathfrak{F}$: The stress (σ, μ) in $\mathfrak{R}$ satisfies

$$(\sigma, \mu; \sigma^*, \mu^*) = 0 \tag{28}$$

where (σ^*, μ^*) is any stress in $\mathfrak{F}$. This condition can be rewritten as

$$\iiint_V [\sigma^{ij}(u^*_{i,j}-\psi^*_{ij})+\mu^{ijk}\psi^*_{ij,k}]\,\mathrm{d}V$$

$$= -\iiint_V [\sigma^{ij}{}_{,j}u^*_i+(\sigma^{[ij]}+\mu^{ijk}{}_{,k})\psi^*_{ij}]\,\mathrm{d}V$$

$$+\iint_{\partial V} (\sigma^{ij}n_j u^*_i+\mu^{ijk}n_k\psi^*_{ij})\,\mathrm{d}(\partial V) = 0$$

where u^*_i and ψ^*_{ij} correspond to (σ^*, μ^*). Hence the stress (σ, μ) in $\mathfrak{R}$ satisfies the equilibrium condition in the absence of external forces:

$$\begin{aligned} \sigma^{ij}{}_{,j} &= 0, & \sigma^{[ij]}+\mu^{ijk}{}_{,k} &= 0 \quad \text{in } V \\ \sigma^{ij}n_j &= 0, & \mu^{ijk}n_k &= 0 \quad \text{on } \partial V. \end{aligned} \tag{29}$$

This shows that $\mathfrak{N}$ is a subspace of $\mathfrak{R}$. Define a subspace $\mathfrak{E}$ as the intersection of $\mathfrak{C}$ and $\mathfrak{R}$. Therefore, the stress in $\mathfrak{E}$ and the corresponding strain satisfy (21) and (29). It then follows that

Decomposition theorem. Hilbert space $\mathfrak{H}$ is decomposed as

$$\mathfrak{H} = \mathfrak{F} \oplus \mathfrak{E} \oplus \mathfrak{N} = \mathfrak{F} \oplus \mathfrak{R} = \mathfrak{C} \oplus \mathfrak{N} \tag{30}$$

$$\mathfrak{C} = \mathfrak{F} \oplus \mathfrak{E}, \qquad \mathfrak{R} = \mathfrak{N} \oplus \mathfrak{E}. \tag{31}$$

That is, any stress (σ, μ) can be written in the form

$$\begin{aligned} \sigma^{ij} &= \sigma^{ij}_F+\sigma^{ij}_E+\sigma^{ij}_N = \sigma^{ij}_F+\sigma^{ij}_R = \sigma^{ij}_C+\sigma^{ij}_N \\ \mu^{ijk} &= \mu^{ijk}_F+\mu^{ijk}_E+\mu^{ijk}_N = \mu^{ijk}_F+\mu^{ijk}_R = \mu^{ijk}_C+\mu^{ijk}_N \end{aligned} \tag{32}$$

where

$$(\sigma_F, \mu_F) \in \mathfrak{F}, (\sigma_E, \mu_E) \in \mathfrak{E}, (\sigma_N, \mu_N) \in \mathfrak{N}, (\sigma_R, \mu_R) \in \mathfrak{R}, (\sigma_C, \mu_C) \in \mathfrak{C}. \tag{33}$$

By virtue of (32), any stress can be expressed in the form

$$\begin{aligned} \sigma^{ij} &= \sigma^{ij}_C+\sigma^{ij}_N, \qquad \mu^{ijk} = \mu^{ijk}_C+\mu^{ijk}_N \\ \sigma^{ij}_C &= E^{ijpq}(u_{p,q}-\psi_{pq})+F^{ijpqr}\psi_{pq,r} \\ \mu^{ijk}_C &= F^{pqijk}(u_{p,q}-\psi_{pq})+G^{pqrijk}\psi_{pq,r} \\ \sigma^{ij}_N &= \phi^{ijk}{}_{,k}, \qquad \mu^{ijk}_N = -\phi^{[ij]k}+\gamma^{ijkl}{}_{,l} \end{aligned} \tag{34}$$

where

$$\phi^{ijk} = 0, \qquad \gamma^{ijkl} = 0 \quad \text{on } \partial V.$$

Let us assume that the body V is subjected to the prescribed body force K^i and body couple $L^{ij}\,(= -L^{ji})$ per unit volume, and to the prescribed surface force T^i and surface couple $M^{ij}\,(= -M^{ji})$ per unit area. It is also assumed that the expressions on the left-hand side of (21) have the certain values $S_{ijk}\,(= -S_{ikj})$ and $R_{ijkl}\,(= -R_{jikl} = -R_{ijlk})$, which may be called the incompatibility tensors. These conditions can be written as

$$\sigma^{ij}{}_{,j}+K^i = 0, \qquad \sigma^{[ij]}+\mu^{ijk}{}_{,k}+L^{ij} = 0$$
$$\varepsilon_{i[j,k]}+\kappa_{i[jk]} = S_{ijk}, \qquad \kappa_{ij[k,l]} = R_{ijkl}. \tag{35}$$

Substituting σ^{ij} and μ^{ijk} from the general expression (34) into these conditions, we have

$$\left.\begin{array}{ll}\sigma_C^{ij}{}_{,j}+K^i = 0, & \sigma_C^{[ij]}+\mu_C^{ijk}{}_{,k}+L^{ij} = 0 \\ \varepsilon_{Ni[j,k]}+\kappa_{Ni[jk]} = S_{ijk}, & \kappa_{Nij[k,l]} = R_{ijkl}\end{array}\right\} \text{ on } V \tag{36a}$$

$$\left.\begin{array}{ll}\sigma_C^{ij}n_j = T^i, & \mu_C^{ijk}n_k = M^{ij} \\ \phi_N^{ijk} = 0, & \gamma_N^{ijkl} = 0\end{array}\right\} \text{ on } \partial V \tag{36b}$$

where ε_{Nij} and κ_{Nijk} correspond to (σ_N, μ_N).

6. EIGEN STRESS

Let us consider a stress (σ, μ) in $\mathfrak{E}$. The force and moment exerting on a surface in V can also be expressed in the form of (11). Since the left-hand sides of (11) vanish for any surface on ∂V, the following integrals are one-valued functions on $[\partial V]$:

$$\Phi^i(P) = \tfrac{1}{2}\int_A^P \phi^{ijl}e_{jlp}\,\mathrm{d}x^p$$
$$\Gamma^{ij}(P) = \tfrac{1}{2}\int_A^P (x^{[j}\phi^{i]kl}+\gamma^{ijkl})e_{klp}\,\mathrm{d}x^p \tag{37}$$

where A and P are a fixed and a generic point on each continuous portion of $[\partial V]$. It then follows that

$$\left.\begin{array}{l}\phi^{ijk} = \Phi^i{}_{,p}e^{jkp} \\ \gamma^{ijkl} = (\Gamma^{ij}{}_{,p}-x^{[j}\Phi^{i]}{}_{,p})e^{klp}\end{array}\right\} \text{ on } [\partial V]. \tag{38}$$

Let $\partial D_\alpha^+, \partial D_\alpha^-$, and B_α^+, B_α^- be the boundary $\partial[\partial V]$ of $[\partial V]$ corresponding to ∂D_α and B_α, and let $(\partial D_\alpha^+, B_\alpha^+), \ldots$ be the intersecting point of two appropriate curves. Defining the sense of D_α and B_α so that ∂D_α^+ and B_α^+ constitutes a part of the oriented $\partial[\partial V]$,

$$\iint_{D_\alpha} \sigma^{ij}n_j\,\mathrm{d}D_\alpha = \Phi^i(\partial D_\alpha^+, B_\alpha^-)-\Phi^i(\partial D_\alpha^+, B_\alpha^+)$$
$$= \Phi^i(\partial D_\alpha^-, B_\alpha^-)-\Phi^i(\partial D_\alpha^-, B_\alpha^+) \equiv \Xi^i(D_\alpha)$$
$$\iint_{D_\alpha} (x^{[j}\sigma^{i]k}+\mu^{ijk})n_k\,\mathrm{d}D_\alpha = \Gamma^{ij}(\partial D_\alpha^+, B_\alpha^-)-\Gamma^{ij}(\partial D_\alpha^+, B_\alpha^+)$$
$$= \Gamma^{ij}(\partial D_\alpha^-, B_\alpha^-)-\Gamma^{ij}(\partial D_\alpha^-, B_\alpha^+) \equiv \Omega^{ij}(D_\alpha). \tag{39}$$

Let P_α be a point on B_α, and let P_α^+ and P_α^- be the corresponding points on B_α^+ and B_α^-, respectively. It then follows that

$$\Phi^i(P_\alpha^-)-\Phi^i(P_\alpha^+) = \Xi^i(D_\alpha)$$
$$\Gamma^{ij}(P_\alpha^-)-\Gamma^{ij}(P_\alpha^+) = \Omega^{ij}(D_\alpha). \tag{40}$$

Let Q_α be a point on ∂D_α, and let Q_α^+ and Q_α^- be the corresponding points on ∂D_α^+ and ∂D_α^- respectively. We then have

$$\begin{aligned}\Phi^i(\partial D_\alpha^-, B_\alpha^+) - \Phi^i(\partial D_\alpha^+, B_\alpha^+) &= \Phi^i(Q_\alpha^-) - \Phi^i(Q_\alpha^+) \equiv \Xi_I^i(B_\alpha) \\ \Gamma^{ij}(\partial D_\alpha^-, B_\alpha^+) - \Gamma^{ij}(\partial D_\alpha^+, B_\alpha^+) &= \Gamma^{ij}(Q_\alpha^-) - \Gamma^{ij}(Q_\alpha^+) \equiv \Omega_I^{ij}(B_\alpha).\end{aligned} \tag{41}$$

The stress (σ, μ) in $\mathfrak{E}$ have the following two alternative expressions:

$$\begin{aligned}\sigma^{ij} &= E^{ijpq}(u_{p,q} + \psi_{pq}) + F^{ijpqr}\psi_{pq,r} \\ &= \phi^{ijk}{}_{,k} \\ \mu^{ijk} &= F^{pqijk}(u_{p,q} + \psi_{pq}) + G^{pqrijk}\psi_{pq,r} \\ &= -\phi^{[ij]k} + \gamma^{ijkl}{}_{,l}.\end{aligned}$$

Now the inner product of stresses in $\mathfrak{E}$ becomes as

$$\begin{aligned}(\sigma, \mu; \sigma^*, \mu^*) &= \iiint_V [\phi^{ijk}{}_{,k}\varepsilon_{ij}^* + (-\phi^{[ij]k} + \gamma^{ijkl}{}_{,l})\kappa_{ijk}^*]\, \mathrm{d}V \\ &= -\iiint_V [\phi^{ijk}(\varepsilon_{i[j,k]}^* + \kappa_{i[jk]}^*) + \gamma^{ijkl}\kappa_{ij[k,l]}^*]\, \mathrm{d}V \\ &\quad + \iint_{\partial V} (\phi^{ikl}\varepsilon_{ik}^* + \gamma^{ijkl}\kappa_{ijk}^*)n_l\, \mathrm{d}(\partial V).\end{aligned}$$

Using the above expression (38) for ϕ^{ikl} and γ^{ijkl}, we have

$$\begin{aligned}(\sigma, \mu; \sigma^*, \mu^*) &= \iint_{\partial V} [\Phi^i{}_{,p}\varepsilon_{ik}^* + (\Gamma^{ij}{}_{,p} - x^j\Phi^i{}_{,p})\kappa_{ijk}^*]e^{klp}n_l\, \mathrm{d}(\partial V) \\ &= \int_{\partial[\partial V]} [\Phi^i(\varepsilon_{ik}^* - x^j\kappa_{ijk}^*) + \Gamma^{ij}\kappa_{ijk}^*]\, \mathrm{d}x^k \\ &\quad - \iint_{[\partial V]} [\Phi^i(\varepsilon_{i[k,p]}^* - x^j\kappa_{ij[k,p]}^* - \kappa_{i[pk]}^*) + \Gamma^{ij}\kappa_{ij[k,p]}^*]e^{klp}n_l\, \mathrm{d}(\partial V) \\ &= \sum_\alpha \Big\{ \Xi^i(D_\alpha)\int_{B_\alpha} (\varepsilon_{ik}^* - x^j\kappa_{ijk}^*)\, dx^k + \Omega^{ij}(D_\alpha)\int_{B_\alpha} \kappa_{ijk}^*\, \mathrm{d}x^k \\ &\quad - \Xi_I^i(B_\alpha)\int_{\partial D_\alpha} (\varepsilon_{ik}^* - x^j\kappa_{ijk}^*)\, \mathrm{d}x^k - \Omega_I^{ij}(B_\alpha)\int_{\partial D_\alpha} \kappa_{ijk}^*\, \mathrm{d}x^k \Big\} \\ &= \sum_\alpha [\Xi^i(D_\alpha)\xi_i^*(B_\alpha) + \Omega^{ij}(D_\alpha)\omega_{ij}^*(B_\alpha)].\end{aligned} \tag{42}$$

It can easily be seen that the macro-stress and couple-stress are given as functions of $\xi_i(B_\alpha)$ and $\omega_{ij}(B_\alpha)$. Therefore, the inner product $(\sigma, \mu; \sigma^*, \mu^*)$ is given as a bi-linear form of $\xi_i(B_\alpha)$, $\omega_{ij}(B_\alpha)$, and $\xi_i^*(B_\alpha)$, $\omega_{ij}^*(B_\alpha)$. Hence the dimension of $\mathfrak{E}$ is $6p$.

7. CLASSICAL THEORY OF ELASTICITY

The classical elasticity is obtained as the limiting case where the rigidity for couple-stress becomes infinity. As G_{ijkpqr} tends to infinity, μ^{ijk} goes to zero, because κ_{ijk} should remain finite. In such a case, κ_{ijk} is independent of σ^{ij} and μ^{ijk}, $\sigma^{[ij]}$ vanishes, and σ^{ij} is a symmetric tensor. Accordingly, U becomes as

$$U = \tfrac{1}{2}E_{(ij)(kl)}\sigma^{ij}\sigma^{kl} \tag{43}$$

where parentheses used with indices indicate the mean with respect to indices:

$$E_{(ij)kl} = \tfrac{1}{2}(E_{ijkl}+E_{jikl}). \tag{44}$$

From (14) and (23), it follows that

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon_{(ij)} = \frac{\partial U}{\partial \sigma^{ij}} = u_{(i,j)} \\ \psi_{ij} &= u_{[i,j]}. \end{aligned} \tag{45}$$

Eliminating u_i from the expression for ε_{ij}, we have the well-known condition of compatibility in the small for the classical theory of elasticity:

$$\varepsilon_{[i[j,k]l]} = 0. \tag{46}$$

Introducing $\mu^{ijk} = 0$ in (7b), we have

$$\phi^{[ij]k} = \gamma^{ijkl}{}_{,l}.$$

Hence it follows that

$$\phi^{(ij)k} = \phi^{[ik]j} + \phi^{[jk]i} = \gamma^{ikjl}{}_{,l} + \gamma^{jkil}{}_{,l}.$$

Therefore, the expression for σ^{ij} becomes as

$$\sigma^{ij} = \sigma^{(ij)} = \Psi^{ikjl}{}_{,kl} \tag{47}$$

where Ψ^{ikjl} is the stress function for the classical theory of elasticity;

$$\Psi^{ikjl} = \Psi^{jlik} = -\Psi^{kijl} = \gamma^{ikjl} + \gamma^{jkil}. \tag{48}$$

8. CONCLUSION

In this paper, the decomposition of Hilbert space $\mathfrak{H}$, in which stress distributions are expressed as points, was derived. Any macro-stress and couple-stress are expressed by a sum of two parts: one is derived from macro-displacements and micro-rotations, and the other from stress functions. The general properties of eigen stress were also discussed. All the characteristics of $\mathfrak{H}$ are almost the same as those for the classical theory of elasticity. The present intrinsic theory was developed on the basis of the global condition of equilibrium as the classical theory of elasticity.

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Абстракт—В настоящей работе определяется линейная, свойственная теория для континуума Коссера в смысле дальнейшего развития классической теории упругости. Путем введения функций напряжения, выводится теорема разложения пространства Гильберта для распределения напряжений. Обсуждаются свойства так наз. собственного напряжения.